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Spacetime Coarse Grainings and the Problem of Time in the Decoherent Histories Approach to Quantum Theory

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Abstract We investigate the possibility of assigning consistent probabilities to sets of histories characterized by whether they enter a particular subspace of the Hilbert space of a closed system during a given time interval. In particular we investigate the case that this subspace is a region of the configuration space. This corresponds to a particular class of coarse grainings of spacetime regions. We consider the arrival time problem, as a warm up, to deal with the problem of time in reparametrization invariant theories (as for example in canonical quantum gravity) which subsequently we examine. Decoherence conditions and probabilities for those application are derived. The resulting decoherence condition does not depend on the explicit form of the restricted propagator that was problematic for generalizations such as application in quantum cosmology. Closely related to our results, is the problem of tunnelling time as well as the quantum Zeno effect. Some interpretational comments conclude, and we discuss the applicability of this formalism to deal with the arrival time problem and the problem of time in general.

Keywords Decoherent (consistent) histories · Problem of time · Arrival time · Restricted propagator

1 Introduction

1.1 Opening Remarks

Questions that involve time in a non-trivial way are not easily addressed within the context of quantum mechanics. An alternative formulation of standard quantum theory that is suited to deal with such questions is the decoherent histories approach. In particular we will deal with the probability that a system is found in (or crosses) a spacetime region using the

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decoherent histories approach to quantum theory. In the present paper we will focus in the case that this region has the simple form of a region of the configuration space extended in the time direction i.e. $\Delta \times [t, t_0]$ with Δ being a region of the configuration space and $[t, t_0]$ a (parameter) time interval. Both the arrival time problem and the problem of time in reparametrization invariant theories, which we are going to analyze, are closely tied with this question.

1.2 The Decoherent Histories Approach

We will briefly review the decoherent histories approach in nonrelativistic quantum theory described by a Schrödinger equation [1–16]. This approach is designed to deal with closed system where there is no separation between observer and system. The (predictive) statements are made about histories of the system and it is well suited to deal with questions that involve time non trivially, i.e. not only about one-time propositions. The central object in this approach is the decoherence functional.

$$D(\underline{\alpha}, \underline{\alpha}') = \mathrm{Tr}(C_{\underline{\alpha}} \rho C^{\dagger}_{\underline{\alpha}'}) \tag{1}$$

with $C_{\underline{\alpha}}$, being the class operator that corresponds to a history $\underline{\alpha}$. In particular, $C_{\underline{\alpha}}$ is given by a string of time-ordered projectors at times $t_1 \dots t_n$,

$$C_{\underline{\alpha}} = P_{\alpha_n}(t_n) \cdots P_{\alpha_2}(t_2) P_{\alpha_1}(t_1) \tag{2}$$

and $\underline{\alpha}$ denotes the string $\alpha_1, \alpha_2, \dots, \alpha_n$. The projection operators are in the Heisenberg picture,

$$P_{\alpha_k}(t_k) = e^{iHt_k} P_{\alpha_k} e^{-iHt_k} \tag{3}$$

where the projectors form a complete and orthogonal set, i.e.

$$\sum_{\alpha} P_{\alpha} = \mathbf{1} \tag{4}$$

and

$$P_{\alpha} P_{\beta} = \delta_{\alpha\beta} P_{\alpha}. \tag{5}$$

This class operator defined in (2) obeys the following

$$\sum_{\underline{\alpha}} C_{\underline{\alpha}} = \mathbf{1}. \tag{6}$$

Note that we could have used an alternative definition of the class operator that in the case of standard nonrelativistic quantum mechanics lead to the same probabilities, namely

$$C_{\underline{\alpha}} = P_{\alpha_n} e^{-iH(t_n - t_{n-1})} P_{\alpha_{n-1}} \cdots e^{-iH(t_2 - t_1)} P_{\alpha_1} \tag{7}$$

which satisfies

$$\sum_{\underline{\alpha}} C_{\underline{\alpha}} = e^{-iH(t_n - t_1)}. \tag{8}$$

The distinction of those two is trivial in nonrelativistic quantum mechanics but not so in reparametrization-invariant theories as was noted in [17] and we will use both definitions in what follows. We may assign probabilities to the possible histories provided that we have a set of those such that

$$\mathrm{Re}\, D(\underline{\alpha}, \underline{\alpha}') = 0 \tag{9}$$

for $\underline{\alpha} \neq \underline{\alpha}'$. While this condition guarantees that the probability rules will be obeyed (including the additivity of disjoint regions of the sample space) we will focus on a stronger condition, namely that both the real and imaginary part of the off-diagonal terms of the decoherence functional vanish, i.e.

$$D(\underline{\alpha}, \underline{\alpha}') = 0 \tag{10}$$

for $\underline{\alpha} \neq \underline{\alpha}'$. This is related with the existence of generalized records [3] as well as considerations of composite systems [18].

Finally, in the case of set of histories obeying the decoherence condition, we may assign probabilities to histories, taking the diagonal terms of the decoherence functional

$$p(\alpha_1, \alpha_2, \ldots) = D(\underline{\alpha}, \underline{\alpha}) = \mathrm{Tr}(C_{\underline{\alpha}} \rho\, C^{\dagger}_{\underline{\alpha}}). \tag{11}$$

To summarize, in order to deal with a question in the decoherent histories, we require first to construct a class operator that correspond to this question. Then make sure that the histories decohere. The latter can be done either by introducing some interacting environment or by considering special initial states. In either case, the system has to obey a decoherence condition. Finally, if the decoherence condition is satisfied, we assign probabilities to the histories corresponding to the diagonal terms of the decoherence functional.

1.3 Arrival Time Problem

The first issue that we will consider is the arrival time problem. This is the question "*what is the probability of a particle entering* (*crossing*) *a region* Δ *of space at any time between* t_1 *and* t_2". This question involves time in a non-trivial way. It is a *spacetime* question. Standard quantum mechanics has difficulties with this kind of question, since all the propositions refer to one particular time. Time has a strange status in quantum mechanics since there is no self-adjoint operator representing it and appears like a parameter [19–21]. Closely related to this are also questions of tunnelling time (how much time the particle spends in the forbidden region) as well as in reparametrization invariant theories, where there is no time (see below). The literature on propositions in quantum mechanics that refer to more than one time is long [22–28]. To deal with this, people have tried to define time in quantum mechanics as an operator [29–31], or using physical internal clocks [19, 20] or using path integral approaches [21, 32–38]. In this paper we will use the decoherent histories approach as in [39].[1] Similar considerations were made in [40].

[1] This paper studies the arrival time problem using class operators of the form of (16) and included an environment with the aim of obtaining decoherence of histories. It was claimed that for most initial system states, approximate decoherence and probabilities consistent with intuition can be obtained. The claim of decoherence is in fact false since, as noted in Sect. 3.1, these coarse grainings are too strong for decoherence, allowing no leakage of probability. The error was due to the use of heuristic path integral methods which obscured the key property (28) of the restricted propagator. However, this paper is correct in its essential conclusions if it is re-interpreted as an implementation of the suggestion of Sect. 3.4—that the class operators consist of time-ordered products of projectors acting at a large but finite number of discrete moments of time.

1.4 Problem of Time

The questions referring to spacetime regions, as in the arrival problem, relates with the so-called "problem of time". This arises in theories that have vanishing Hamiltonian. They are called reparametrization invariant theories, since anything observable is independent of time, in the sense that redefining the (parameter) time will leave all the probabilities invariant. The most notorious of these theories arises in quantum cosmology, where the wavefunction of the universe obeys the Wheeler-De Witt equation,

$$H\Psi[h_{ij}, \phi] = 0. \tag{12}$$

The wavefunction Ψ depends on the three-metric h_{ij} and the matter field configurations ϕ on a closed three-surface [41–43]. Other examples are the parametrized particle and the relativistic particle. In all these cases, the lack of time parameter forces us to make statements concerning regions of the configuration space, without singling out one variable as time. This will be easier to carry out if we have an effective way to treat spacetime questions in the context of quantum mechanics as, for example, in the arrival time case.

There are two particular approaches that have made progress in the quantization of these "timeless" theories. The one is the evolving constants method [45–52]. The other that we are going to follow here, is the decoherent histories (for previous related work see [17, 44, 53, 54]). In particular the requirements are the following:

(a) The state $|\psi\rangle$ should obey the constraint,

$$H|\psi\rangle = 0. \tag{13}$$

(b) In the context of a timeless theory, one needs to think afresh what a class operator is. The lack of time forces us to drop the simple definition (2). In particular we require the class operators to commute with the constraint (Hamiltonian).

$$[C_\alpha, H] = 0. \tag{14}$$

Note also, that the class operator C_α, is a projector "classically". This means that if all the projectors at different (parameter) time commuted, the class operator would reduce to a projector.

(c) The inner product in the decoherence functional should be the so-called "induced inner product" for the states to be normalized [55–61]. This is needed because the states for most of these theories, are not normalized in the usual Schrödinger inner product. This is effectively an inner product defined on the solution surface. See [44, 53, 62] for applications similar to those considered here.

Of course, on top of these, as in all the cases in decoherent histories, we require decoherence in order to be able to assign probabilities. This latter part is examined in detail in this paper. Note that in Sect. 4 we will focus on the case that the initial state is an energy eigenstate, rather than having vanishing Hamiltonian (which is just a special case for $E = 0$). This would correspond in shifting the total energy by a constant amount. The class operator will still need to commute with the Hamiltonian and the only difference is that the constraint will then be $(H - E)|\psi\rangle = 0$. It will help us seeing some easy non-trivial examples, since for most simple models the zero energy eigenstate is trivial. The latter is clearly not true for the cosmology case that was mentioned above.

1.5 This Paper

This paper examines the decoherent histories approach when applied the arrival time problem and the problem of time in reparametrization invariant theories. These correspond to a particular class of spacetime questions. In particular in Sect. 2 we will review different forms of the restricted propagator that is of use, and derive a new expression. In Sect. 3 we will consider the arrival time problem. First in Sect. 3.1 we will get a general decoherence condition and we will explore it in detail in Sect. 3.2, analyze some examples in Sect. 3.3, and discuss the consequences for the probabilities in Sect. 3.4. In Sect. 4 we will deal with the problem of time in reparametrization invariant theories and provide a general decoherence condition. This condition is independent of the detailed calculation of the restricted propagator and depends solely on the energy eigenfunctions and the boundary of the region of interest. In Sect. 4.2 we will also suggest what we can calculate in the case of quantum cosmology, following the suggested formalism. We summarize and conclude in Sect. 5.

2 New Expression for the Restricted Propagator

An important mathematical object in all the discussion of the arrival time in the decoherent histories, as well as in many other applications, is the *restricted propagator*. This is the propagator restricted to some particular region Δ (of the configuration space) that corresponds to a subspace of the total Hilbert space denoted by $\mathcal{H}_\Delta$. In this section we will provide several different expressions including a new one and prove their equivalence. The most common form is the path integral one:

$$g_r(x,t\mid x_0,t_0)=\int_\Delta \mathcal{D}x\exp(iS[x(t)])=\langle x|g_r(t,t_0)|x_0\rangle. \tag{15}$$

The integration is done over paths *that remain in the region* Δ *during the time interval* $[t,t_0]$. The $S[x(t)]$ is as usual the action. Note that if there is ambiguity about which region is the propagator restricted to we will add a superscript (say $g_r^\Delta(t,t_0)$ for example). The operator form of the above is given by [17, 63]:

$$g_r(t,t_0)=\lim_{\delta t\to 0}Pe^{-iH(t_n-t_{n-1})}P\cdots Pe^{-iH(t_1-t_0)}P. \tag{16}$$

With $t_n=t$, $\delta t\to 0$ and $n\to\infty$ simultaneously keeping $\delta t\times n=(t-t_0)$. H is the Hamiltonian operator. P is a projection operator on the restricted region Δ and in cases that it is not clear we will add a subscript (say P_Δ). We therefore have

$$g_r(x,t\mid x_0,t_0)=\langle x|g_r(t,t_0)|x_0\rangle. \tag{17}$$

Note here that the expression (16) is the defining one for cases that the restricted region is not a region of the configuration space, but some other subspace of the total Hilbert space $\mathcal{H}$. The differential equation obeyed by the restricted propagator is:

$$\left(i\frac{\partial}{\partial t}-H\right)g_r(t,t_0)=[P,H]g_r(t,t_0). \tag{18}$$

Which is almost the Schrödinger equation, differing by the commutator of the projection to the restricted region with the Hamiltonian.

We will show that the restricted propagator can also be expressed as

$$g_r(t,t_0) = P\exp\left(-i(t-t_0)PHP\right)P. \tag{19}$$

This relation will turn out to be the most useful in our paper. Note that PHP is the Hamiltonian projected in the subspace $\mathcal{H}_\Delta$. To prove (19) we multiply (18) with P we will then get

$$\left(i\frac{\partial}{\partial t} - PHP\right)g_r(t,t_0) = 0 \tag{20}$$

using the fact that $P[H,P]P = 0$ and that the propagator has a projection P at the final time. This is Schrödinger equation with Hamiltonian PHP. It is evident that this leads to the full propagator in $\mathcal{H}_\Delta$ provided that the operator PHP is self-adjoint in this subspace [67]. We will now prove the equivalence of (19) with (16). Taking the limit $\delta t \to 0$ implies:

$$\begin{aligned}
g_r(t,t_0) &= \lim_{\delta t\to 0} P(1-iH\delta t)P(1-iH\delta t)\cdots(1-iH\delta t)P \\
&= \lim_{\delta t\to 0}\left\{P + (-i\delta t n)PHP + (-i\delta t)^2\binom{n}{2}PHPHP + \cdots\right\} \\
&= \lim_{\delta t\to 0}\left\{P + (-i\delta t PHP) + (-i\delta t PHP)^2\binom{n}{2}\right. \\
&\quad \left. + \cdots + (-i\delta t PHP)^k\binom{n}{k} + \cdots\right\}.
\end{aligned} \tag{21}$$

Noting that in the limit we consider, that $\delta t \to 0$ and $n \to \infty$ keeping $\delta t \times n = t - t_0$:

$$(\delta t)^k\binom{n}{k} \to \frac{(\delta t n)^k}{k!}. \tag{22}$$

We have:

$$g_r(t,t_0) = \lim_{\delta t\to 0}\left\{P + (-i\delta t PHPn) + \cdots + \frac{(-i\delta t PHPn)^k}{k!} + \cdots\right\} \tag{23}$$

and therefore we get:

$$\begin{aligned}
g_r(t,t_0) &= P\exp\{-i(t-t_0)PHP\}P \qquad (24)\\
&= P\exp\{-i(t-t_0)HP\}. \qquad (25)
\end{aligned}$$

The above expressions will be used in the following sections. Finally, note that (19) satisfies trivially (18).

3 Arrival Time Problem in the Decoherent Histories

From here on we will be using the decoherent histories approach to quantum theory. Let the configuration space $\mathcal{Q}$ be a (Riemannian) manifold that for simplicity we will assume to be $\mathbb{R}^n$ and consider a region Δ in that. The question that we will be asking is "*What is the probability that the system crosses the region* Δ *within the time interval* $[t,t_0]$". The most natural way to deal with this is to ask the probability that the system remains during this time interval, in the complimentary region $\bar{\Delta}$ and deduce the answer to our question by classical logic (the negation is the answer to the arrival time question). To apply classical logic we require the histories to decohere. We will proceed to find what condition the initial state and the time interval should satisfy for the latter to be true. Note that the region $\bar{\Delta}$ is defined as $\Delta \cup \bar{\Delta} = \mathcal{Q}$ and $\Delta \cap \bar{\Delta} = \emptyset$.

3.1 Decoherence Condition and Probabilities

The class operator corresponding to the history of remaining in $\bar{\Delta}$ is

$$C_{\bar{\Delta}} = g_r^{\bar{\Delta}}(t, t_0) \tag{26}$$

where the restricted region of propagation is $\bar{\Delta}$ corresponding to projector $\bar{P}$ (from here on, when referring to restricted propagator, it will be understood that it is in the region $\bar{\Delta}$ and we will thus omit the superscript). The class operator for entering the region Δ in that time interval is equal with the crossing propagator

$$C_{\Delta} = g_c(t, t_0) = g(t, t_0) - g_r(t, t_0) \tag{27}$$

where $g(t, t_0)$ is the full propagator. The above definition of the crossing propagator follows directly from the path integral expressions, since the paths that cross region Δ are all the paths except those that remain always in $\bar{\Delta}$. This class operator, is also consistent with (8). We will assume for simplicity pure initial state $|\psi\rangle$. If the restricted Hamiltonian $H_r = \bar{P} H \bar{P}$ is self-adjoint operator in $\mathcal{H}_{\bar{\Delta}}$, and using (19) we have

$$g_r^{\dagger}(t, t_0) g_r(t, t_0) = \bar{P}. \tag{28}$$

We should point out that H_r is indeed guaranteed to be self-adjoint in $\mathcal{H}_{\bar{\Delta}}$ in the following two cases

(i) The subspace spanned by $\bar{P}$ is finite dimensional. This relates to the usual account of the quantum Zeno effect [64–66].
(ii) The subspace spanned by $\bar{P}$ is a region $\bar{\Delta}$ in the configuration space and the Hamiltonian is quadratic in momenta. This has been shown in [67] (see also [68]).

The problems that we will deal with fall in this second class. To have decoherence, and thus being able to assign probabilities to histories, we require that the off-diagonal terms of the decoherence functional vanish.

$$D(\Delta, \bar{\Delta}) = \langle \psi | C_{\Delta}^{\dagger} C_{\bar{\Delta}} | \psi \rangle = 0. \tag{29}$$

Using (28) this leads to the very general condition

$$\langle \psi | g^{\dagger}(t, t_0) g_r(t, t_0) | \psi \rangle = \langle \psi | \bar{P} | \psi \rangle \tag{30}$$

where $g^{\dagger}(t, t_0) = e^{iH(t-t_0)}$ is the full propagator. This is the main result of this section and will be explored in detail in the next section. For this condition to be satisfied, we will show that the initial state $|\psi\rangle$ has to vanish on the boundary of the region $\bar{\Delta}$ that we will denote as $\partial \bar{\Delta}$, that is

$$\langle x | \psi \rangle = 0, \quad \forall x \in \partial \bar{\Delta}. \tag{31}$$

Note that while the vanishing boundary for the initial state is a necessary condition for decoherence, it is by no means a sufficient one. We will analyze the consequences of (30) for the general case in the next section. Here we will give a heuristic argument that the state vanishing on the boundary is a necessary condition. The point is that for the state

$$g_r(t,t_0)|\psi\rangle = |\psi(t)\rangle_r \tag{32}$$

to exist, i.e. belongs in the Hilbert space $\mathcal{H}$ we require the state $|\psi\rangle$ to vanish on the boundary. To see this note that the Hamiltonian involves a term with ∂^2. The projector on a region corresponds to a characteristic function $\chi_{\bar{\Delta}}(x)$ defined as

$$\chi_{\bar{\Delta}}(x) = \begin{cases} 1, & x \in \bar{\Delta}, \\ 0, & x \in \Delta. \end{cases} \tag{33}$$

The state $\bar{P}|\psi\rangle$ involved in $|\psi(t)\rangle_r$ will be discontinuous on the boundary unless $|\psi\rangle$ obeys (31). It follows that the state $|\psi(t)\rangle_r$ will not be square integrable and will not belong to $\mathcal{H}$ if $|\psi\rangle$ does not vanish on the boundary due to the delta function that will appear from the derivative of a discontinuous function. For more detail the reader is referred at the Appendix.

Given that we have initial state and time interval such that the histories C_Δ and $C_{\bar{\Delta}}$ decohere, we can assign them probabilities. The probability for no-entering the region Δ is

$$p_r = D(\bar{\Delta}, \bar{\Delta}) = \langle\psi|\bar{P}|\psi\rangle \tag{34}$$

and the crossing probability

$$p_c = 1 - p_r = \langle\psi|P|\psi\rangle. \tag{35}$$

An important thing to note, is that if we had an initial state belonging to the subspace $\mathcal{H}_{\bar{\Delta}}$, i.e.

$$\bar{P}|\psi\rangle = |\psi\rangle \tag{36}$$

the crossing probability would be zero. This means that for a state localized outside the region Δ, the probability that it crosses the region Δ is zero. This is less problematic than it may seems in the first sight, since most states will not decohere and therefore the above "candidate-probabilities" would not correspond to real probabilities. We will return to this point in Sect. 3.4.

3.2 A Detailed Look at the Decoherence Condition

We will explore the consequences of (30). The time evolved stated $|\psi(t)\rangle$ is

$$|\psi(t)\rangle = \exp(-i\hat{H}(t-t_0))|\psi\rangle = g(t,t_0)|\psi\rangle \tag{37}$$

and the restricted time-evolved state is given by (32). The decoherence condition requires that the overlap of those two states at time t is the same as their overlap at the initial time t_0,

$$\langle\psi(t)|\psi(t)\rangle_r = \langle\psi|\bar{P}|\psi\rangle. \tag{38}$$

This will in general depend (usually sensitively) on the time t. It is physically reasonable to require the decoherence to persist in time and not depend sensitively on the time interval.

The restricted time-evolved state corresponds to considering the region of restriction with same Hamiltonian and introducing infinite potential walls on the boundary. The state would then be reflected on the boundary. The decoherence condition (30) is satisfied in the following four cases. The first three of those are independent of the time interval. In the next section we will give one distinct simple example of each of those.

(a) The state $|\psi\rangle$ is an energy eigenstate.[2]

$$H|\psi\rangle = E|\psi\rangle. \tag{39}$$

On top of this it has to vanish on the boundary (31). If the latter is true, then the state $|\psi\rangle$ is a restricted energy eigenstate, i.e.

$$H_r\bar{P}|\psi\rangle = E\bar{P}|\psi\rangle \tag{40}$$

with the same energy as the initial state. This will be explored in detail in Sect. 4.2. It implies that the restricted time-evolved state is given by

$$|\psi(t)\rangle_r = e^{-iE(t-t_0)}\bar{P}|\psi\rangle. \tag{41}$$

The full time-evolved state is of course

$$|\psi(t)\rangle = e^{-iE(t-t_0)}|\psi\rangle \tag{42}$$

and therefore their overlap satisfies the decoherence condition (30).

(b) The restricted propagator can be expressed by the use of method of images. Then there will be a class of states that decohere. The restricted propagator can be written as

$$g_r(t,t_0) = \bar{P}e^{-i\hat{H}(t-t_0)}\left(\mathbf{1} + \sum_n \alpha_n \mathbf{R}_n\right)\bar{P} \tag{43}$$

with $\mathbf{R}_n$ corresponding to reflection due to each image and α_n is the weight of the image. By considering the fact that at $t = t_0$ we should get $\bar{P}$, we have the following condition on the images

$$\bar{P}\sum_n \alpha_n \mathbf{R}_n \bar{P} = 0. \tag{44}$$

The requirement we get to have decoherence is

$$\left(\mathbf{1} + \sum_n \alpha_n \mathbf{R}_n\right)\bar{P}|\psi\rangle = |\psi\rangle. \tag{45}$$

In that case we have

$$|\psi(t)\rangle = g(t,t_0)|\psi\rangle = e^{-i\hat{H}(t-t_0)}\left(\mathbf{1} + \sum_n \alpha_n \mathbf{R}_n\right)\bar{P}|\psi\rangle \tag{46}$$

[2] This will be of great interest in the reparametrization invariant case, where all physical states are necessarily energy eigenfunctions.

which implies

$$\langle\psi|g_r^\dagger(t,t_0)|\psi(t)\rangle = \langle\psi|g_r^\dagger(t,t_0)g_r(t,t_0)|\psi\rangle \tag{47}$$

and therefore the condition (30) is satisfied. There are many states obeying (45) and we can easily generate them. Take a fiducial state $|\chi\rangle$ such that

$$\bar{P}|\chi\rangle = |\chi\rangle. \tag{48}$$

This means that the state is localized in the region $\bar{\Delta}$. Note that the fiducial state vanishes on the boundary due to the continuity of the wavefunction. We define

$$\mathbf{R}_n|\chi\rangle = |\phi_n\rangle. \tag{49}$$

We can see that

$$\bar{P}\sum_n \alpha_n|\phi_n\rangle = \bar{P}\sum_n \alpha_n R_n \bar{P}|\chi\rangle = 0. \tag{50}$$

This implies that the following state obeys trivially (45).

$$|\psi\rangle = |\chi\rangle + \sum_n \alpha_n|\phi_n\rangle. \tag{51}$$

It is therefore possible to construct a state $|\psi\rangle$ that decoheres for every fiducial state $|\chi\rangle$ that is for every state localized in the $\bar{\Delta}$ region. This is still a very limited class of initial states that decohere. Moreover, the initial state is not localized in the restricted region.

We should point out here that restricted propagator can be written as a function of the full propagator, using the method of images, if and only if there exist a set of energy eigenstates, vanishing on the boundary, that when projected on the region Δ forms a dense subset of the subspace $\mathcal{H}_\Delta$, i.e. span $\mathcal{H}_\Delta$. This is equivalent with requiring that the restricted energy spectrum (aka spectrum of the restricted Hamiltonian H_r) is a subset of the (unrestricted) energy spectrum. The latter is not in general the case. An extreme example where the two spectrums are completely different is the following.

Consider a particle in an infinite potential well (box) with width say $(d-a)$. Let us consider the restricted region $\bar{\Delta}$ being a subinterval say $[b,c]$ such that $(b-c)$ is not a rational multiple of $(d-a)$. The (non-trivial) restricted energy spectrum of the region is totally different from the (unrestricted) energy spectrum of the full potential well.

(c) The full time-evolution (for this time interval) of the initial state $|\psi\rangle$ does not leave the subspace $\mathcal{H}_{\bar{\Delta}}$ i.e. remains always in $\bar{\Delta}$. In that case it is evident that the restricted and the full propagator coincide and thus condition (30) is trivially satisfied.

(d) Recurrences. The full time-evolved state of the system happens to return to same overlap with the restricted time-evolved state as initially. This is in general very sensitive in time, and thus of no physical significance. There is however a case that the decoherence, while still time dependent, may persist for a small time interval. We could have a state $|\psi\rangle$ that for small t the full time evolution (unitary) is within the region $\bar{\Delta}$ as in case (c) above, but in a later time leaves the region. If this state, further in the future, has a recurrence, i.e. the full and restricted propagator coincide again, the decoherence will persist, at least for a while, until the full evolution crosses the region Δ.

Before proceeding to examples of the above cases, we should point out what happens in the physically interesting case that the initial state is localized in $\bar{\Delta}$. The time persistent cases

(a) and (b) do not apply. We may get decoherent histories that persist in time ONLY in case (c) where the system is not supposed to cross the region Δ anyway. This will be discussed in Sect. 3.4.

3.3 Examples

We will see here four simple examples situations where there are initial states that decohere. Each of those corresponds to one of the above mentioned cases. Note though, that these are simple examples to illustrate the cases analyzed above, and they definitely do not exhaust the physical interesting examples that belong to the cases analyzed in the previous subsection. Moreover, the conclusions reached about the decoherence properties of the proposed class operators that are discussed in the next section, do *not* rely on the particular examples of this section.

(a) Consider a simple harmonic oscillator with Hamiltonian

$$H=\frac{1}{2}\left(x^2-\frac{\partial^2}{\partial x^2}\right) \tag{52}$$

where we have taken $\hbar=m=\omega=1$. Let the initial state $|\psi\rangle$ be an energy eigenstate and in particular the $n=2$ corresponding to energy $E=5/2$. This is given by

$$\langle x|\psi\rangle=A(2x^2-1)\exp\left(-\frac{x^2}{2}\right) \tag{53}$$

with $A=\frac{\sqrt{2}}{2}\left(\frac{1}{\pi}\right)^{1/4}$ normalization constant. We want to ask what is the probability that the system crosses the region Δ. For a general region Δ this history would not decohere. If we choose the region in such a way that the initial wavefunction vanishes on the boundary of this region, then according to Sect. 3.2 the histories would decohere. We therefore choose the region Δ to be the (closed) interval $[-\sqrt{2}/2,\sqrt{2}/2]$. The probability to cross this region Δ is then given by

$$p_c=|A|^2\int_{\sqrt{2}/2}^{-\sqrt{2}/2}(2x^2-1)^2\exp(-x^2)dx\approx 0.2. \tag{54}$$

Note also that in this example the restricted propagator cannot be written with the method of images, since the spectrum of the restricted Hamiltonian does not coincide with the full Hamiltonian spectrum. It just shares (at least) one common eigenvalue.

(b) Consider a free particle in two dimensions with Hamiltonian

$$H=\hat{\mathbf{p}}^2. \tag{55}$$

We take the region Δ to be outside a wedge of angle β where $\beta=\pi/b$ with b being an even integer. The restricted propagator for the region $\bar{\Delta}=\{\mathbf{x}\in\mathbb{R}^2\mid 0\leq\theta\leq\beta\}$ is given by the methods of images [69][3]

$$g_r(t,t_0)=\bar{P}e^{-iH(t-t_0)}\sum_{n=0}^{b-1}(\mathbf{R}_n-\mathbf{K}_n)\bar{P} \tag{56}$$

[3]For further interesting results on the wedge problem, see [70].

where $\bar{P}$ is the projection in the wedge, and we introduce the rotation operators

$$\mathbf{R}_n = \int r dr d\theta |r, 2n\beta + \theta\rangle\langle r, \theta|, \tag{57}$$

$$\mathbf{K}_n = \int r dr d\theta |r, 2n\beta - \theta\rangle\langle r, \theta|. \tag{58}$$

Consider now an initial state

$$\psi(r, \theta) = \sin(b\theta) R(r) \tag{59}$$

where $R(r)$ is any function of r such that the $|\psi\rangle$ is normalized. It is clear that this state obeys the decoherence condition (45). The crossing probability will then be

$$p_c = \langle\psi|P|\psi\rangle = \frac{\int_{\pi/b}^{2\pi} \sin^2(b\theta) d\theta}{\int_0^{2\pi} \sin^2(b\theta) d\theta} = 1 - \frac{\beta}{2\pi} \tag{60}$$

as expected.

(c) Here we will consider a bound system, that therefore its full time-evolution remains within a distance from the centre. In particular we will consider the Hamiltonian of the first example (52). The initial state though will not be an energy eigenstate but instead a superposition of two. Let us consider the state

$$\psi(x) = B_1 \exp\left(-\frac{x^2}{2}\right) + B_2(2x^2 - 1) \exp\left(-\frac{x^2}{2}\right) \tag{61}$$

that is superposition of the state with energy $1/2$ with the ground state with energy $5/2$ and $B_1 = \sqrt{2}/2$, $B_2 = 1/2$ i.e. such that the total state is normalized. The two states are with equal weight. The region will be $\bar{\Delta} = [-10, 10]$. We can see that the restricted propagator is approximately the same as the full propagator, since the system is localized in the region $\bar{\Delta}$. The width of the Gaussian is two that is much less than the region we consider. We can therefore see that at least for small time intervals the off-diagonal terms (approximately) vanish. The crossing probability in this case is zero. Note that more complicated examples of this type, could include some interacting environment that restricts the state in $\bar{\Delta}$ for the time interval in question.

(d) We will now consider a particle in an infinite potential well of width 2π. The Hamiltonian is

$$H = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + V(x) \tag{62}$$

where

$$V(x) = \begin{cases} 0, & -\pi \le x \le \pi, \\ \infty, & x < -\pi,\ x > \pi. \end{cases} \tag{63}$$

The region $\bar{\Delta}$ that we will consider is $0 \le x \le \pi$. The particle in an infinite potential well is a periodic system. If the width of a well is α then the energies are

$$E_n = \frac{1}{2}\left(\frac{\pi n}{\alpha}\right)^2. \tag{64}$$

From this we can see that $t = \frac{4\alpha^2}{\pi}$ is the period of the system. The restricted propagator corresponds to a free particle evolution in a well of width $\alpha = \pi$ while the full propagator is in a well of width $\alpha = 2\pi$. Let us consider the case that the time interval is

$$T = t - t_0 = 16\pi. \tag{65}$$

We note that given initial state $|\psi\rangle$ the restricted evolved state at time $t = 16\pi$ is

$$|\psi(t)\rangle_r = \bar{P}|\psi\rangle \tag{66}$$

while the full evolved state

$$|\psi(t)\rangle = |\psi\rangle. \tag{67}$$

From this is evident that the decoherence condition (30) is satisfied. We should note that this depends crucially on the choice of the time interval.

3.4 Discussion

An important question is what happens if the initial state is localized in $\bar{\Delta}$. From (35) we get that the crossing probability is zero (provided we have decoherence and can assign probabilities). This result can be in disagreement with classical intuition. Take for example a free-particle wavepacket localized in positive x-axis, with negative average momentum and ask the question whether it should cross the region of negative x-axis. Classical intuition would tell us that the particle would cross the region of negative x-axis. According to our analysis, the particle would either remain always in the positive x-axis or the above histories would not decohere and we would not be able to make a prediction.

Here we should stress that the results of Sect. 3.1 are independent of the Hamiltonian and therefore apply when the Hamiltonian includes an *environment*. In that case, the projections at each moment of time are of the form $P_{sys} \bigotimes \mathbf{1}_{env}$. The addition of environment, does modify the "reduced" dynamics, but most importantly, it does *not* (in general) produce decoherence for histories characterized by class operators of the form of (16). This would seem natural e.g. [39] since the addition of environment (quantum Brownian motion) is known to bring decoherence of the different (coarse grained) trajectories (e.g. [71]). However the particular coarse graining considered here (spacetime) would *not* allow for decoherence (and thus a definite answer) even for a system coupled with environment such as quantum Brownian motion.[4] This turns out to be consistent with the results of Hartle (e.g. [21]), who noted that some spacetime coarse grainings are "too strong" for decoherence.

Mathematically what stops us from getting non-trivial probabilities for crossing, is that in the continuous limit of projectors, we have no leakage of probability outside the restricted region. This is related with what is called the quantum Zeno effect, in Copenhagen quantum mechanics [64], where continuous observation (corresponding to continuous projectors) stop the state of a system from evolving outside the subspace of observation.

This still leaves us with the problem of how to get histories that would correspond to the question "*did the system crossed the region* Δ *in this time period*?" and agree with classical intuition. The resolution to this apparent paradox comes from the fact that the same classical questions can be interpreted in quantum mechanics in many different ways. In particular

[4] An exception exist for the case where the environment restricts the full evolution in the region $\bar{\Delta}$ for all the time interval.

instead of following the most natural coarse grainings defined in this section, we could try to pose the arrival time question differently. We could do the following three things:

(a) The restricted propagator we considered was defined by taking some discreet number of times t_n, projecting on the restricted region and then taking the continuum limit. We could rather than taking the continuum limit, use some discreet (but frequent enough), number of projections [72]. In between the projections there will be probability leaking out the region. A decoherence mechanism (environment) should also be introduced. This is what was effectively calculated in Halliwell and Zafiris [39] although this was not stated explicitly.

Note also, that this modification (apart from being less natural) may have problems if we want to consider reparametrization invariant theories (as in Sect. 4). This is because the probability will usually depend on the discretization of the times of measuring (number of t_n's) and therefore will be "cut-off" dependent and fail to be independent of reparametrizations.

(b) We could "soften" the notion of measurement, and use POVM's (Positive Operator Valued Measures) instead of projection operators. These are quasi-projectors and form an overcomplete basis of the Hilbert space. They correspond to projection operators in an enlarged Hilbert space, but when restricted to the one of interest stop being orthogonal. They still span all the space. We expect that using those will allow for probability leak from the restricted region and therefore get non-trivial probabilities for crossing as was considered in Anastopoulos and Savvidou [40].

(c) We could have considered a ("smooth") spacetime region that is not of the form $\Delta \times [t, t_0]$ (rectangular). This kind of coarse grainings were also considered in [21]. In particular in the example of the wavepacket, we could have chosen the region to "follow" the wavepacket, i.e. to be at each time a spatial interval around the centre of the wavepacket and with width greater than the width of the wavepacket at that time. This history would decohere (similarly to the case (c) of Sect. 3.2) and give probability approximately one of remaining in this spacetime region. Since the centre of the wavepacket with negative average momentum moves in the negative x-axis, we can *implicitly* deduce that the system crossed the negative x-axis by using classical intuition. Note though, that still in this case we would not be able to get non-zero crossing the boundary of the region, probability due to the same mathematical restrictions as before. Finally, to apply this to reparametrization invariant theories (as in the following section), more care must be taken when choosing the region in order to have a class operator that obeys the constraint.

To conclude, we point out that we do not get any contradiction, and the fact that we can get an answer from one coarse graining while no answer (i.e. no decoherence) from another is due to the many inequivalent ways we can pose the same classical question in quantum mechanics. Here we could stress once again, that the class operators proposed include all situations where we would expect the system to cross the region in question. However, the requirement of decoherence fails for those cases. The inclusion of *any* environment does not change the picture.

4 Problem of Time in the Decoherent Histories Approach

We will now move to another related issue, the problem of models with reparametrization invariance. The main motivation for considering such systems comes from the quantum cosmology and the notorious problem of time in quantum gravity. Finding observables in such theories has been proven a difficult task. As was mentioned in the introduction, there are several approaches to this problem. The one that we follow here is with the use of decoherent histories, and in particular, following the proposal by Halliwell and Wallden in [17] for the construction of class operators that obey the (Hamiltonian) constraint. The question that we want to answer is "*did our system cross the region* Δ *of the configuration space with no reference in time*". This type of questions, form a general enough set of observables (provided that most of them can be assigned a probability). In the decoherent histories analysis, we first have to construct the class operators corresponding to the set of physical questions we want to address. The second part, is to make sure that these class operators decohere and we can assign probabilities, at least for a general enough set of initial states (or regions Δ), even if the coupling of environment is needed. This second part was not considered in [17] and it is the focus of this section in the light of the results we have from the arrival time case and the new form of the restricted propagator. Note also, that the connection with the arrival time problem, is mathematically evident, because of the use of restricted propagators in the definition of the class operators in both cases.

4.1 Decoherence Condition and Probabilities

Inspired by classical considerations, Halliwell and Wallden in [17] defined class operators that are consistent with the constraint and correspond to not crossing the region Δ with no reference in time. This was based on the observation that a whole classical trajectory is reparametrization invariant and the proposed class operator was a continuous infinite temporal product of Heisenberg picture projections operators. This is expressed as,

$$C_{\bar{\Delta}} = \prod_{t=-\infty}^{\infty} \bar{P}(t) = \lim_{t\to\infty, t_0\to-\infty} e^{iHt} g_r(t,t_0) e^{-iHt_0}. \tag{68}$$

The class operator for crossing is

$$C_{\Delta} = 1 - C_{\bar{\Delta}}. \tag{69}$$

The limits in (68) exist when the class operator acts on certain class of states and some of these cases were examined in [17]. It will turn out that for states that obey the decoherence condition, this limit does exist. The decoherence condition, requires the off-diagonal term to vanish and gives

$$\langle\psi|C_{\bar{\Delta}}|\psi\rangle = \langle\psi|C_{\bar{\Delta}}^{\dagger} C_{\bar{\Delta}}|\psi\rangle \tag{70}$$

for a pure state $|\psi\rangle$ that is solution to the constraint equation (i.e. energy eigenstate, or in the case of problem of time, zero energy eigenstates). Given that $|\psi\rangle$ is energy eigenstate with energy E the right hand side of (70) becomes

$$\lim_{t,t'\to\infty, t_0,t_0'\to-\infty} e^{iE(t_0-t_0')} \langle\psi|g_r^{\dagger}(t,t_0) e^{-iH(t-t')} g_r(t',t_0')|\psi\rangle. \tag{71}$$

Provided the individual limits exist (i.e. the class operator is well defined) we can take the t, t' limits simultaneously (and similarly the t_0, t_0' limits). Using (28) this leads to

$$\langle\psi|C_{\bar{\Delta}}^{\dagger} C_{\bar{\Delta}}|\psi\rangle = \langle\psi|\bar{P}|\psi\rangle = p_r \tag{72}$$

which is also the probability for not crossing, provided that we have decoherence (for those states). The left hand side of (70) becomes

$$\langle\psi|C_{\bar{\Delta}}|\psi\rangle = \lim_{t\to\infty, t_0\to-\infty} e^{iE(t-t_0)}\langle\psi|g_r(t,t_0)|\psi\rangle \tag{73}$$

where E is the eigenvalue of the energy eigenstate $|\psi\rangle$. We also have, using (19),

$$\langle\psi|g_r(t,t_0)|\psi\rangle = \langle\psi|\bar{P}|\psi\rangle + \sum_{n=1}^{\infty}\frac{(-i(t-t_0))^n}{n!} \times \left\{\langle\psi|(\bar{P}H)^n|\psi\rangle + \langle\psi|(\bar{P}H)^{n-1}\bar{P}[H,\bar{P}]|\psi\rangle\right\}. \tag{74}$$

Here, we shall assume that the state of our system obeys the following condition

$$\bar{P}[H,\bar{P}]|\psi\rangle = 0. \tag{75}$$

This will turn out to be the decoherence condition, since we will show that it is a sufficient condition to obey (70). With initial state obeying (75), (74) becomes

$$\langle\psi|g_r(t,t_0)|\psi\rangle = \langle\psi|\bar{P}|\psi\rangle\exp(-i(t-t_0)E) \tag{76}$$

due to the fact that $(\bar{P}H)^n|\psi\rangle = E^n\bar{P}|\psi\rangle$ for these states. Using (76) the limits in the class operator in (73) become trivial, since the dependence on t and t_0 disappears. It is now clear that the class operator acting on states that obey this condition is well defined. We therefore get

$$\langle\psi|C_{\bar{\Delta}}|\psi\rangle = \langle\psi|\bar{P}|\psi\rangle. \tag{77}$$

We can now see, that from (72) and (77) the decoherence condition (70) is satisfied. The assumption made about the initial states was (75) which turns out to be the necessary and sufficient condition to have decoherence for histories corresponding to the class operators of the form (68). In the case of decoherence, we can finally, assign the probability given by (72) for histories that do not cross while for histories that cross the probability is

$$p_c = \langle\psi|P|\psi\rangle. \tag{78}$$

4.2 A Detailed Look on the Decoherence Condition

Let us first, have a closer look on the decoherence condition (75). It gives

$$\bar{P}H\bar{P}(\bar{P}|\psi\rangle) = E(\bar{P}|\psi\rangle) \tag{79}$$

where E is the energy of the state $|\psi\rangle$. This is the general result. We require that there exists a state in $\mathcal{H}_{\bar{\Delta}}$ that is restricted energy eigenfunction having the same energy eigenvalue with the total energy of the state $|\psi\rangle$ and that vanishes on the boundary. Examples of this sort were discussed in [17]. Note here that following Appendix for the condition (79) to be satisfied we require a state $|\psi\rangle$ such that

$$\langle x|\psi\rangle = 0, \quad \forall x \in \partial\bar{\Delta}. \tag{80}$$

The vanishing boundary conditions is a necessary condition for the sate to obey condition (79) and therefore decohere. In the reparametrization invariant case, that we discuss here, we restrict our attention to initial states $|\psi\rangle$ that obey the constraint, i.e. are energy eigenstates with energy E determined by the constraint. If we restrict our attention to those states, the condition (79) reduces to the requirement that the wavefunction $|\psi\rangle$ vanishes on the boundary, i.e. the latter is a sufficient condition as well. Here we will show the latter, that proves the claim made in Sect. 3.2 at the (a) case. We have

$$H|\psi\rangle = E|\psi\rangle \tag{81}$$

therefore

$$\bar{P}H(\bar{P}+P)|\psi\rangle = E\bar{P}\psi\rangle \tag{82}$$

which implies

$$\bar{P}H\bar{P}|\psi\rangle + \bar{P}[H,P]|\psi\rangle = E\bar{P}|\psi\rangle. \tag{83}$$

This implies that the state $\bar{P}|\psi\rangle$ is a restricted energy eigenstate with energy E, provided that

$$\bar{P}[H,P]|\psi\rangle = 0. \tag{84}$$

This is indeed the case if and only if the state $|\psi\rangle$ vanishes on the boundary. For a quadratic in momentum Hamiltonian, and with Δ being a region of the configuration space we get

$$[H,P] \propto \hat{p}\delta(x-a) + \delta(x-a)\hat{p} \tag{85}$$

for all $a \in \partial\Delta$. We therefore have $\bar{P}[H,P]|\psi\rangle = 0$ in the case that the state vanish on the boundary and thus (79) is satisfied. Another way to see that this is indeed the case is the following. To find the restricted energy eigenfunctions we need to solve the Schrödinger equation subject to vanishing conditions on the boundary. A total energy eigenfunction that vanishes on the boundary, clearly obeys the restricted Schrödinger equation and therefore is a restricted energy eigenstate. Note that the converse is not true. We could have other wavefunctions being restricted energy eigenstates and not total energy eigenstates but we restrict our attention to the total energy eigenstates since these are the states that obey the constraint.

We can now say that for the reparametrization invariant theories a necessary and sufficient condition to have decoherence, i.e. to obey (79), is that *the state $|\psi\rangle$ has to vanish on the boundary of the region* i.e. satisfy the condition (80). This is the main result of this section. This condition is very restrictive in quantum cosmological models. To this end, we should say that the following two things can be done in order to meet this conditions.

(i) For a given state-solution of the Wheeler-DeWitt equation we can ask what is the locus of the zero's of that wavefunction. From this we can deduce for which regions we can ask whether or not the universe was in, by considering a closed region. This is difficult in general since the solutions are not known, but it can be done for mini-superspace models at least in the semi-classical (WKB) approximation.
(ii) We can do the converse. Given a particular region, we may search for solutions of the Wheeler-DeWitt equation which have this region as locus of their zero's.

4.3 Arrival Time-Reparametrization Relation

In the arrival time problem, if we had restricted our attention on energy eigenstates (case (a) in Sect. 3.2) we would come to the same condition as in the reparametrization invariant case. This is indeed expected, since in that case, the decoherence condition as well as the probability of crossing are both independent of the time interval. Extending the interval from minus to plus infinity is not going to change something.

5 Summary and Conclusions

We have discussed the possibility of assigning consistent probabilities to histories of the system being in a particular region of the configuration space continuously and complementary to this, the probability of crossing a region of the configuration space. Our analysis relies on the decoherent histories approach to quantum theory. The questions considered, correspond to a particular class of spacetime coarse grainings. The main focus of the paper was to get the conditions that the initial state should satisfy (given dynamics of system and environment) in order to have decoherence in the above mentioned types of coarse grainings. We first considered the arrival time problem applying the most natural coarse graining related. Using the new expression for the restricted propagator (19), we derived the general decoherence condition (30). Only in those cases we may assign consistently probabilities in the decoherent histories analysis. We investigated the consequences of this condition and found that very few initial states satisfy it. Moreover, those histories gave always zero crossing probability which came to apparent contradiction with [39] (but see also footnote 1). We discussed this and suggested that the above coarse grainings are "too strong" to allow decoherence in the arrival time questions. This implies that we will have to deal with this problem alternatively implementing the idea that the same classical question correspond to many inequivalent quantum ones. In particular we suggested considering projecting on this region at discreet time intervals (as opposed to the continuous that we used), weakening the notion of measurement and using POVM's instead of projectors and finally using spacetime coarse grainings that are not of the form $\Delta \times [t, t_0]$. Note though, that from those suggestions, only the POVM's could be consistent with the considerations of the next section that dealt with reparametrization invariant class operators.

We then proceeded by considering a closely related issue, that of reparametrization invariance. The use of the decoherent histories for dealing with this aspect of the problem of time is widely considered as promising. In particular we considered the set of reparametrization invariant class operator defined in [17] and in the light of the results we had for the arrival time problem we examined the decoherence properties of the suggested class operator that was not considered previously. We then concluded to a very simple general decoherence condition (79) that is independent of the precise details of the restricted propagator contrary to what was previously believed. This condition turns out to be that the initial state, which is an energy eigenfunction in order to satisfy the constraint, has to vanish on the boundary of the region of the configuration space in question. This is a necessary and sufficient condition. Unfortunately this condition is very restrictive and the decoherent class operators may not cover a general enough set of questions to deal fully with the problem of reparametrization invariant questions. This completes, at least in formal level, the analysis of the proposal of [17] to deal with the problem of reparametrization invariance. Finally, we suggested what can be calculated further using these results in the case of quantum cosmology, but the explicit calculation was left for a future work.

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Appendix: Vanishing Boundary Conditions

Here we will explore what the decoherence conditions (30) and (79) tells us about the boundary conditions. First we should point out that the choice of whether the boundary $\partial\Delta$ belongs to Δ or $\bar{\Delta}$ is irrelevant. That is true because the Hilbert space $P_{\mathrm{cl}}\mathcal{H} = \mathcal{H}_{\Delta\mathrm{cl}}$ corresponding to projection in a closed region is the same with $P_{\mathrm{op}}\mathcal{H} = \mathcal{H}_{\Delta\mathrm{op}}$ that corresponds to an open region. The reason is that the physical states, members of $\mathcal{H}$ $(= L^2[\mathbb{R}])$ are defined as equivalence classes of "almost everywhere" equal functions. Note that two functions f, g are almost everywhere equal, if there are equal everywhere except to measure zero subsets. We therefore have in the norm of the Hilbert space $\|f - g\| = 0$.

Consider a state $|\psi\rangle$ that does not vanish on the boundary. The (unormalized) state $P|\psi\rangle$ will have divergent energy expectation value. Strictly speaking the Hamiltonian operator is not defined on those states since its domain is $\mathrm{AC}^2[\mathbb{R}]$, that is the set of functions in $L^2[\mathbb{R}]$ whose weak derivatives are in $\mathrm{AC}[\mathbb{R}]$. ($\mathrm{AC}[\mathbb{R}]$ is the set of absolutely continuous functions whose weak derivatives are in $L^2[\mathbb{R}]$.) It could not therefore be defined on $P|\psi\rangle$ if the state does not vanish on the boundary. We could still think of expressing it as the limit of a sequence of states belonging at $\mathcal{H}_{\bar{\Delta}}$ (that vanish on the boundary and form a dense subset) where the state converges to $\psi(x)$ everywhere except at the boundary that the sequence vanishes. We can see that states with more and more terms in this sequence have higher and higher energy and can claim that in the limit it will eventually have infinite energy. This was considered as introducing instantaneously a hard wall potential at a point that the wavefunction does not vanish, first by Berry in [73] as well as by Bender et al. in [74]. The evolution of these states was shown to be unitary, but resulting to fractal wavefunction with infinite energy.

In the arrival time decoherence condition (30) we had the overlap of the full evolved state with the restricted evolved state. The latter would correspond to introducing instantaneously an infinite potential and its evolution as was noted in [73, 74] leads to a fractal wavefunction. This means a wavefunction that is zero at most points, but explodes to infinity at some zero (or small) measure intervals. Clearly the overlap of any of these states with the full evolved state that would be a reasonably well behaved state, is close to zero. This means that it is *not* equal with $\langle\psi|\bar{P}|\psi\rangle$ as it has to be in order to decohere.

Let us now consider the condition (79) for the reparametrization invariant case.

$$\bar{P}H\bar{P}|\psi\rangle = E\bar{P}|\psi\rangle. \tag{86}$$

This corresponds to an eigenfunction equation in the subspace $\mathcal{H}_{\bar{\Delta}}$. If the energy is finite, i.e. $E \neq \infty$ we need to have $|\psi\rangle$ vanishing on the boundary, since otherwise we would get infinite energy. Note that for the derivation of the above we used the fact that

$$g_r^{\dagger}(t, t_0)g_r(t, t_0) = \bar{P} \tag{87}$$

which is the case when acting to states that follow unitary evolution (i.e. reversible). As it was analyzed in the above mentioned papers [73, 74] the sates do indeed evolve unitarily.

Let us illustrate the above with a simple example. Let the region $\bar{\Delta}$ be $[0, \pi]$ and $H = \hat{p}^2$.

$$u_n(x) = \begin{cases} \sin nx, & x \in [0, \pi], \\ 0, & x \in (-\infty, 0) \cup (\pi, \infty). \end{cases} \tag{88}$$

These are a complete set of states (a dense subset) of $L^2[0,\pi]$. Let the state $P|\psi\rangle = |f\rangle$ and $\langle x|f\rangle = f(x)$. We can therefore write

$$f(x) = \sum_{n=1}^{\infty} f_n u_n(x). \tag{89}$$

This series will converge on $f(x)$ if $f(0) = f(\pi) = 0$ everywhere. Otherwise it will converge to $f(x)$ everywhere except at 0 and π. The average energy of this state will go as

$$\langle f|H|f\rangle \propto \sum_{n=1}^{\infty} |f_n|^2 n^2. \tag{90}$$

This series will diverge unless f_n vanishes sufficiently quicker than n^2 as n goes to infinity. It can been shown, that for this to be the case $f(0)$ and $f(\pi)$ should both vanish. In the case they do not, the steeper the slope (i.e. more terms in the series meaning closer to the actual function), the bigger the energy of the state. It can thus be argued, that requiring the introduction of a rigid wall "instantaneously", i.e. using exact projections, would lead to an infinite energy state.

To conclude, we noticed that the states that do not vanish on the boundary evolve to fractal wavefunctions and therefore their overlap with the full evolved state does not obey the condition (30). These states also have infinite (average) energy in the subspace $\mathcal{H}_{\bar{\Delta}}$ and therefore fail to satisfy the decoherence condition (79) as well.

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